1. Since, f, g unif. cont. on A. Since f, g bounded, $\exists M_1, M_2$ s.t. $|f(x)| \le M_1$, $g(x) \le M_2$ $\forall x \in A$

Let $\varepsilon > 0$, $\exists \delta_1 > 0$ s.t. if $x, u \in A$, $|x-u| < \delta_1$

$$|f(x) - f(u)| < \varepsilon / 2M_2$$

$\exists \delta_2 > 0$ s.t. if $x, u \in A$, $|x-u| < \delta_2$

$$|g(x) - g(u)| < \varepsilon / 2M_1$$

Take $\delta = \min\{\delta_1, \delta_2\}$. if $x, u \in A$, $|x-u| < \delta$,

$$\begin{aligned} |(fg)(x) - (fg)(u)| &= |f(x)g(x) - f(x)g(u) + f(x)g(u) - f(u)g(u)| \\ &\le |f(x)||g(x) - g(u)| + |f(x) - f(u)||g(u)| \\ &\le M_1 |g(x) - g(u)| + M_2 |f(x) - f(u)| \\ &< \varepsilon \end{aligned}$$

Then fg is unif. cont. on A.

Can not drop e.g. $f(x) = x$, $g(x) = \sin x$.

2. Assume not. i.e. f is not bounded.

$\forall n \in \mathbb{N}$, $\exists x_n \in A$ s.t. $|f(x_n)| > n$

Since A is bounded, (x_n) is bounded.

B-W, $\Rightarrow$ $\exists$ subsequence (x_{n_k}) of (x_n) s.t. (x_{n_k}) converges.

Then (x_{n_k}) is Cauchy in A

(Thm 5.4.7) $\Rightarrow$ $f(x_{n_k})$ is Cauchy

$\Rightarrow$ $f(x_{n_k})$ is bounded

Contradicting to $f(x_{n_k}) > n_k > k$ $\forall k \in \mathbb{N}$

Then f is bounded.

3. Since f is cont. and I closed bounded interval

f attain min. in I.

$\exists\, t \in I$ s.t. $f(t) = \alpha$, α is min of f in I

Also, f is uniformly cont. on I.

let $\varepsilon > 0$, $\exists\, \delta > 0$ s.t. if $x, u \in I$, $|x-u| < \delta$

$$|f(x) - f(u)| < \varepsilon \cdot \alpha^2$$

choose above δ, if $x, u \in I$, $|x-u| < \delta$

$$\left| \frac{1}{f(x)} - \frac{1}{f(u)} \right| = \frac{|f(x) - f(u)|}{|f(x)||f(u)|} \leq \frac{|f(x) - f(u)|}{\alpha^2} < \varepsilon$$

Then $\frac{1}{f}$ is uniformly cont. on I.

Can not drop: $f = x^2$, $I = (0, \infty)$

4. let $\varepsilon > 0$, $\exists\, K \in \mathbb{R}$ s.t. if $x \geq K$, $|f(x) - \ell| < \frac{\varepsilon}{2}$

Let $I = [0, K+1]$, closed bounded Interval

Then I is uniformly cont. on I.

i.e. $\exists\, \delta_I > 0$ s.t. if $x, u \in I$, $|x-u| < \delta_I$, $|f(x) - f(u)| < \varepsilon$

let $J = [K, \infty)$, if $x, u \in J$

$$|f(x) - f(u)| \leq |f(x) - \ell| + |f(u) - \ell| < \varepsilon$$

Choose $\delta = \min\{\delta_I, 1\}$, Clearly $[0, \infty) = I \cup J$

if $x, u \in [0, \infty)$, $|x-u| < \delta$,

Case 1: $x < K$, $\Rightarrow$ $u < K+1 \Rightarrow x, u \in I \Rightarrow |f(x) - f(u)| < \varepsilon$

Case 2: $K < x < K+1$ $\Rightarrow$ $x \in I$ and $x \in J$ $\Rightarrow$ done

case 3: $x > K+1$ $\Rightarrow$ $u > K \Rightarrow x, u \in J \Rightarrow |f(x) - f(u)| < \varepsilon$.

5. Since if $x \in \mathbb{R}$, $x = kp + y$, for some $k \in \mathbb{N}$, $y \in [0, p)$

Then for $I = [0, p]$, f is continuous on I.

$\Rightarrow$ f is bounded on I

$|f(y)| < M$ for all $y \in [0, p]$

for all $x \in \mathbb{R}$,

$|f(x)| = |f(kp+y)| = |f(y)| < M$.

Then f is bounded on $\mathbb{R}$.

Consider $J = [-1, p+1]$ closed bounded interval.

f is unif. cont. on J.

Let $\varepsilon > 0$, $\exists\ \delta_J$ s.t. if $x, u \in J$, $|x-u| < \delta_J$, $|f(x) - f(u)| < \varepsilon$

Choose $\delta = \min\{1, \delta_J, p\}$

If $x, u \in \mathbb{R}$, $|x-u| < \delta$

let $x = kp + y$, $k \in \mathbb{N}$, $y \in [0, p)$

since $|x-u| < \delta$, let $u = kp + z$

Then $|y - z| < \delta \Rightarrow z \in [-1, p+1)$

Then $|f(y) - f(z)| < \varepsilon$

since $f(x) = f(y)$, $f(u) = f(z)$

$|f(x) - f(u)| = |f(y) - f(z)| < \varepsilon$

f is uniformly cont. on $\mathbb{R}$.